Frame Equivalence
and
Weak Equivalence Principle

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$f(R)$ gravity & its representations

The action of the $f(R)$ theory:

$$S\left[g_{\mu\nu},\Psi_m\right]=\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}f(R)+S_m\left(g_{\mu\nu},\Psi_m\right)$$

Representation in **Jordan frame**:

$$S_{\mathcal{J}}\left[\phi,g_{\mu\nu},\Psi_m\right]=\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[\phi R-V(\phi)\right]+S_m\left(g_{\mu\nu},\Psi_m\right)$$

$$\tilde{\phi}=\sqrt{\frac{3}{2}}\frac{1}{\kappa}\ln\phi=\sqrt{\frac{3}{2}}\frac{1}{\kappa}\ln f'(R)$$

$$\tilde{g}_{\mu\nu}=\phi g_{\mu\nu}=f'(R)g_{\mu\nu}$$

Representation in **Einstein frame**:

$$S_{\mathcal{E}}\left[\tilde{\phi},\tilde{g}_{\mu\nu},\Psi_m\right]=\int d^4x\sqrt{-\tilde{g}}\left[\frac{1}{2\kappa^2}\tilde{R}-\frac{1}{2}\left(\tilde{\nabla}\tilde{\phi}\right)^2-\tilde{V}(\tilde{\phi})\right]$$

$$+S_m\left(e^{-\sqrt{2/3}\kappa\tilde{\phi}}\tilde{g}_{\mu\nu},\Psi_m\right)$$

Which frame is physically correct?

As a generic aspect of any scalar-tensor theory, two frames are available to describe the BD theory. One frame is called the Jordan frame (JF) in which the BD field equations were originally written and the BD scalar field played the role of a spin-0 component of gravity. The other is the conformally rescaled Einstein frame (EF) in which the scalar field plays the role of a source matter field. There is a long standing debate as to whether the descriptions of the BD theory in the two frames, JF and EF, should be considered physically eqivalent. In order to get a flavor of this debate and the resulting confusion, we should only say that physicists are divided roughly into six groups depending on their attitude to the question. They can be listed as follows. Some authors: (1) neglect the issue, (2) think that the two frames are physically equivalent, (3) consider them physically nonequivalent but do not provide supporting arguments, (4) regard only JF as physical but, if necessary, use EF for mathematical convenience, (5) regard only EF as physical, (6) belong to two or more of the above categories!

from Bhadra *et. al.*, MPLA 22(2007)367

Which frame is physically correct?

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Mach's Principle and Invariance under Transformation of Units*

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A gravitational theory compatible with Mach's principle was published recently by Brans and Dicke. It is characterized by a gravitational field of the Jordan type, tensor plus scalar field. It is shown here that a coordinate-dependent transformation of the units of measure can be used to throw the theory into a form for which the gravitational field appears in the conventional form, as a metric tensor, such that the Einstein field equation is satisfied. The scalar field appears then as a "matter field" in the theory. The invariance of physical laws under coordinate-dependent transformations of units is discussed.

In a recent paper,[1] a modified relativistic theory of gravitation, closely related to Jordan's theory,[2] was developed, compatible with Mach's principle. It was indicated that the resulting formalism was but one particular representation of the theory, based upon a particular definition of the units of mass, length, and time.

The purposes of this note are, first to discuss very briefly the invariance of physical laws under units transformations,[3] and second to give another representation of the above theory, completely equivalent to it and derived from it by a simple transformation of units.

The first representation of the theory[1] could be characterized concisely as a relativity theory for which must be invariant under a transformation of units. (The units and dimensions employed need not be three in number, nor need they be limited to the traditional mass, length, and time.)

The invariance which we wish to consider here is broader than the elementary consideration described above. Imagine, if you will, that you are told by a space traveller that a hydrogen atom on Siruis has the same diameter as one on the earth. A few moments' thought will convince you that the statement is either a definition or else meaningless. It is evident that two rods side by side, stationary with respect to each other, can be intercompared and equality established in the sense of an approximate congruence between them. However,

Which frame is physically correct?

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The conformal transformation's controversy: what are we missing?

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An alternative interpretation of the conformal transformations of the metric is discussed according to which the latter can be viewed as a mapping among Riemannian and Weyl-integrable spaces. A novel aspect of the conformal transformation's issue is then revealed: these transformations relate complementary geometrical pictures of a same physical reality, so that, the question about which is the physical conformal frame, does not arise. In addition, arguments are given which point out that, unless a clear statement of what is understood by "equivalence of frames" is made, the issue is a semantic one. For definiteness, an intuitively "natural" statement of conformal equivalence is given, which is associated with conformal invariance of the field equations. Under this particular reading, equivalence can take place only if the metric is defined up to a conformal equivalence class. A concrete example of a conformal-invariant theory of gravity is then explored. Since Brans-Dicke theory is not conformally invariant, then the Jordan's and Einstein's frames of the theory are not equivalent. Otherwise, in view of the alternative approach proposed here, these frames represent complementary geometrical descriptions of a same phenomenon. The different points of view existing in the literature are critically scrutinized on the light of the new arguments.

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Conformal frames are physically indistinguishable

The conformal transformation:

$$g^{\mathcal{J}}_{\mu\nu}(\mathbf{x_A}) \quad \longrightarrow \quad g^{\mathcal{E}}_{\mu\nu}(\mathbf{x_A}) = \Omega^2\left(x^\alpha\right) g^{\mathcal{J}}_{\mu\nu}(\mathbf{x_A})$$

- The conformal factor Ω is position-dependent
- **Physics must be invariant under a choice of the units**
- Rescaling the units of length, time, and mass is a conformal transformation
- Frames $\mathcal{J}$ and $\mathcal{E}$ are physically equivalent provided that the units of physical quantities are allowed to scale with Ω^n in $\mathcal{E}$
- We keep the units fixed in $\mathcal{J}$ while admit them to run in $\mathcal{E}$
- **Conformal transformations are rescalings of units**, i.e. **frame transformations**, **NOT coordinate transformations**

Running units, an example

- The cosmological redshift is the shift in the spectra of the distant light sources in the universe when compared with the spectra of the light sources on the Earth:

$$1 + z \equiv \frac{\lambda_0}{\lambda_\oplus}$$

where λ_0 is the wavelength of a spectral line of the distant source observed on the Earth, and $\lambda_\oplus$ is that of the source on the Earth.

Chi, Gu & Lee (2013)

Running units, an example

- Consider a Robertson-Walker metric in the Jordan frame:

$$ds^2 = -a^2(\eta)\left(-d\eta^2 + dx^2 + dy^2 + dz^2\right)$$

and consider the conformal transformation,

$$\tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x)\,, \quad \Omega(x) = 1/a(\eta)$$

that transforms the Robertson-Walker metric in the Jordan frame to a Minkowskian metric in the Einstein frame:

$$d\tilde{s}^2 = -d\eta^2 + dx^2 + dy^2 + dz^2$$

Running units, an example

- In the Jordan frame, due to the expansion of the universe, the wavelength of a light will be redshifted by a factor

$$\frac{a(\eta_0)}{a(\eta_s)} \equiv 1 + z_g$$

when traveling from the source to the Earth, where η_0 and η_s respectively denote the present time and the time when the light is emitted, and z_g denotes the gravitational redshift. Accordingly,

$$\lambda_0 = [a(\eta_0)/a(\eta_s)]\, \lambda_{\rm em}$$

where $\lambda_{\rm em}$ is the wavelength of the light when emitted.

Running units, an example

- If we further assume that in the Jordan frame the wavelength of a spectral line at the source at the emission time is the same as that on the Earth at present, i.e.,

$$\lambda_{\rm em} = \lambda_\oplus$$

we will obtain

$$\begin{aligned} 1+z &\equiv \lambda_0/\lambda_\oplus \\ &= [a(\eta_0)/a(\eta_s)]\,(\lambda_{\rm em}/\lambda_\oplus) \\ &= a(\eta_0)/a(\eta_s) \equiv 1+z_g \end{aligned}$$

Running units, an example

- In contrast, in the Einstein frame there is no gravitational redshift because the space-time is Minkowskian in this example, and therefore

$$\lambda_0^{\mathcal{E}} = \lambda_{\mathrm{em}}^{\mathcal{E}}$$

- However, there is another redshift caused by the conformal transformation that makes the units in the Einstein frame running (i.e., different units at different space-time points), if we assume the units are fixed in the Jordan frame.

- In this case the units depend on time due to the time-dependent conformal factor, $\Omega(x) = 1/a(\eta)$. In particular, in the Einstein frame the length unit $\ell_u^{\mathcal{E}}$ at different times are related by

$$\ell_u^{\mathcal{E}}(\eta_1)/\ell_u^{\mathcal{E}}(\eta_2) = \Omega(\eta_1)/\Omega(\eta_2) = a(\eta_2)/a(\eta_1)$$

Running units, an example

- Accordingly, the length unit at the emission time and that at present are different. For example, the length 1 meter at the emission time is different from 1 meter at present, and they are related by $1\,\mathrm{m}^{\mathcal{E}}(\eta_{\mathrm{em}}) = [a(\eta_0)/a(\eta_{\mathrm{em}})]\,1\,\mathrm{m}^{\mathcal{E}}(\eta_0)$

- As a consequence, the wavelength of a spectral line of the distant source at the emission time and that of the source on the Earth at present are different and related by

$$\lambda^{\mathcal{E}}_{\mathrm{em}} = [a(\eta_0)/a(\eta_{\mathrm{em}})]\,\lambda^{\mathcal{E}}_{\oplus}$$

and therefore in the Einstein frame

$$\begin{aligned} 1 + z^{\mathcal{E}} &\equiv \lambda^{\mathcal{E}}_0/\lambda^{\mathcal{E}}_{\oplus} \\ &= \lambda^{\mathcal{E}}_{\mathrm{em}}/\lambda^{\mathcal{E}}_{\oplus} \\ &= a(\eta_0)/a(\eta_{\mathrm{em}}) \\ &= 1 + z_g = 1 + z \end{aligned}$$

The line element in Jordan frame can be written as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = a^2(\eta) \left(-d\eta^2 + d\mathbf{x}^2 \right) \tag{1}$$

where the cosmic time t has been recasted in the conformal time η as $dt = a(\eta)\, d\eta$. We assume fixed units for physical observables in this frame, i.e. there is no local changes in units for the Jordan frame.

When Jordan frame experiences a conformal transformation with a position-dependent conformal factor Ω, the line element is governed by

$$ds_{\mathcal{E}}^2 = g_{\mu\nu}^{\mathcal{E}} dx^\mu dx_\nu = a_{\mathcal{E}}^2(\eta) \left(-d\eta^2 + d\mathbf{x}^2 \right) = \Omega^2 g_{\mu\nu} dx^\mu dx^\nu \tag{2}$$

with

$$a^{\mathcal{E}}(\eta) \ = \ \Omega\, a(\eta) \tag{3}$$

$$dt^{\mathcal{E}} \ = \ a^{\mathcal{E}}(\eta)\, d\eta = \Omega\, dt \tag{4}$$

$$d\ell^{\mathcal{E}} \ = \ a^{\mathcal{E}}(\eta) d\mathbf{x} = \Omega\, d\ell \tag{5}$$

Rescaling of Units

Conformal Transformation $\equiv$ Units Rescaling

$$ds_{\mathcal{E}} = \Omega\, ds \quad \Rightarrow \quad \ell_u^{\mathcal{E}} = \Omega\, \ell_u, \;\; t_u^{\mathcal{E}} = \Omega\, t_u \quad \Rightarrow \quad c = 1 \tag{6}$$

If we fix the Planck costant $\hbar = 1$, then

$$\hbar \equiv 1 \quad \Rightarrow \quad m_u^{\mathcal{E}} = \frac{m_u}{\Omega} = \Omega^{-1} m_u \tag{7}$$

Therefore, the conformal factor can actually be viewed as the normalized (w.r.t. $\mathcal{J}$ frame) standard unit scale of length in $\mathcal{E}$ frame, i.e.

$$\Omega = \frac{\ell_u^{\mathcal{E}}}{\ell_u} = \frac{t_u^{\mathcal{E}}}{t_u} = \frac{m_u}{m_u^{\mathcal{E}}} = \frac{T_u}{T_u^{\mathcal{E}}} \tag{8}$$

The last equality holds if we choose the Boltzmann constant $k_{\mathrm{B}} = 1$. The consequence of the local transformation of units (the running units) give rise to

$$\frac{ds_{\mathcal{E}}(\mathbf{x}_A)}{ds_{\mathcal{E}}(\mathbf{x}_B)} = \frac{\Omega(\mathbf{x}_A)}{\Omega(\mathbf{x}_B)} = \frac{a^{\mathcal{E}}(\eta_A)}{a^{\mathcal{E}}(\eta_B)} \tag{9}$$

Measured values of a physical quantity

In light of the above relations, we see that the measured value $\mathcal{X}^{\varepsilon}_{\rm obs}$ of any physical quantity $\mathcal{X}^{\varepsilon}$ in $\mathcal{E}$ is actually a constant regardless the local changes in the associated unit $u^{\varepsilon}_{\mathcal{X}}$, i.e.

$$\mathcal{X}^{\varepsilon} = \mathcal{X}^{\varepsilon}_{\rm obs} u^{\varepsilon}_{\mathcal{X}} \quad \Rightarrow \quad \mathcal{X}^{\varepsilon}_{\rm obs} = \frac{\mathcal{X}^{\varepsilon}(\mathbf{x}_A)}{u^{\varepsilon}_{\mathcal{X}}(\mathbf{x}_A)} = \frac{\mathcal{X}^{\varepsilon}(\mathbf{x}_B)}{u^{\varepsilon}_{\mathcal{X}}(\mathbf{x}_B)} \tag{10}$$

Chi, Gu & Lee (2013)

Redshifts

In JF, we have

$$1 + z = \frac{\lambda_0}{\lambda_{\text{em}}} = \frac{a(\eta_0)}{a(\eta)} = \frac{a_0}{a} \tag{11}$$

In EF,

$$1 + z_{\varepsilon} = \frac{\lambda_0^{\varepsilon}(\eta_0)}{\lambda_{\text{em}}^{\varepsilon}(\eta_0)} = \frac{\ell_0^{\varepsilon}(\eta_0)/\Omega(\mathbf{x}_0)}{\ell_{\text{em}}^{\varepsilon}(\eta_0)/\Omega(\mathbf{x}_0)} = \frac{\ell_0^{\varepsilon}(\eta)\left[a^{\varepsilon}(\eta_0)/a^{\varepsilon}(\eta)\right]}{\ell_{\text{em}}^{\varepsilon}(\eta_0)} = \frac{\ell_0^{\varepsilon}(\eta)\left[a^{\varepsilon}(\eta_0)/a^{\varepsilon}(\eta)\right]}{\ell_{\text{em}}^{\varepsilon}(\eta)\left[\Omega(\mathbf{x}_0)/\Omega(\mathbf{x}_{\text{em}})\right]} \tag{12}$$

But at the emitting time we have $\ell_{\text{em}}^{\varepsilon}(\eta) = \ell_0^{\varepsilon}(\eta)$, we have

$$1 + z_{\varepsilon} = \frac{a^{\varepsilon}(\eta_0)/\Omega(\mathbf{x}_0)}{a^{\varepsilon}(\eta)/\Omega(\mathbf{x})} = \frac{a(\eta_0)}{a(\eta)} = \frac{a_0}{a} = 1 + z \tag{13}$$

Hence, the redshift is a frame-invariant quantity (fiq).

CMB Temperature

In JF, the CMB temperature is governed by

$$T_0 = T(\eta_0) = \frac{\text{const.}}{\lambda_{\text{max}}(\eta_0)} = \frac{\text{const.}}{\lambda_{\text{max}}(\eta)(1+z)} = \frac{T_{\text{em}}(\eta)}{1+z} \tag{14}$$

While in EF,

$$T_0^{\varepsilon} = T(\eta_0) = \frac{\text{const.}}{\lambda^{\varepsilon}_{\text{max}}(\eta_0)} = \frac{\text{const.}}{\lambda^{\varepsilon}_{\text{max}}(\eta)(1+z_{\varepsilon})} = \frac{T^{\varepsilon}_{\text{em}}(\eta)}{1+z_{\varepsilon}} \tag{15}$$

Wein's law remains the same regardless which frame we take.

Distances

Since the comoving distance at redshift z, $r(z)$, is obtained by using the null geodesic $ds^2 = 0$, which is obviously frame-invariant, therefore it is a fiq.

$$r(z) = \int_0^r d\,r = \int_t^{t_0} \frac{d\,t}{a} = \int_{\eta_0}^{\eta} d\,\eta = r^{\mathcal{E}}(z_{\mathcal{E}}) \tag{16}$$

The angular diameter distance:

$$d_A = \frac{a_0\, r}{1+z} = \frac{a_0^{\mathcal{E}}\, r^{\mathcal{E}}}{1+z_{\mathcal{E}}} = d_A^{\mathcal{E}} \tag{17}$$

Likewise, the luminosity distance is also a fiq:

$$d_L = a_0\, r(1+z) = a_0^{\mathcal{E}}\, r^{\mathcal{E}}(1+z_{\mathcal{E}}) = d_L^{\mathcal{E}} \tag{18}$$

Accordingly, the relation between d_A and d_L is also frame-invariant:

$$d_L(z) = d_A(z)(1+z)^2 \tag{19}$$

Hubble parameter

The comoving distance is usually written as

$$r = \int_{\eta_0}^{\eta} d\,\eta = \int_z^0 \frac{d\,z}{d\,z/d\,\eta} = \int_z^0 \frac{d\,z}{\frac{d\,z}{d\,t}\frac{d\,t}{d\,\eta}} = \int_0^z \frac{d\,z}{(1+z)aH}$$

We see that

$$aH = a^\varepsilon H^\varepsilon \quad \Rightarrow \quad H^\varepsilon = H \cdot \left(\frac{a}{a^\varepsilon}\right) = H\Omega^{-1}$$

On the other hand, we can calculate $d\,z/d\,\eta$ in EF according to the fiq. $1+z^\varepsilon$:

$$\frac{d\,z^\varepsilon}{d\,\eta} = (1+z^\varepsilon)\left[\frac{\partial_\eta^\varepsilon \Omega}{\Omega} - \frac{\partial_\eta^\varepsilon a^\varepsilon}{a^\varepsilon}\right]$$

Thus, one has

$$r = \int_{z^\varepsilon}^0 \frac{d\,z^\varepsilon}{d\,z^\varepsilon/d\,\eta} = \int_0^{z^\varepsilon} (1+z^\varepsilon)\left[\frac{\partial_\eta^\varepsilon a^\varepsilon}{a^\varepsilon} - \frac{\partial_\eta^\varepsilon \Omega}{\Omega}\right]$$

Consequently,

$$H^\varepsilon = \frac{\partial_\eta^\varepsilon a^\varepsilon}{a^\varepsilon} - \frac{\partial_\eta^\varepsilon \Omega}{\Omega} = \frac{\partial_t^\varepsilon\, a}{a} \neq \frac{\partial_t^\varepsilon\, a^\varepsilon}{a^\varepsilon}$$

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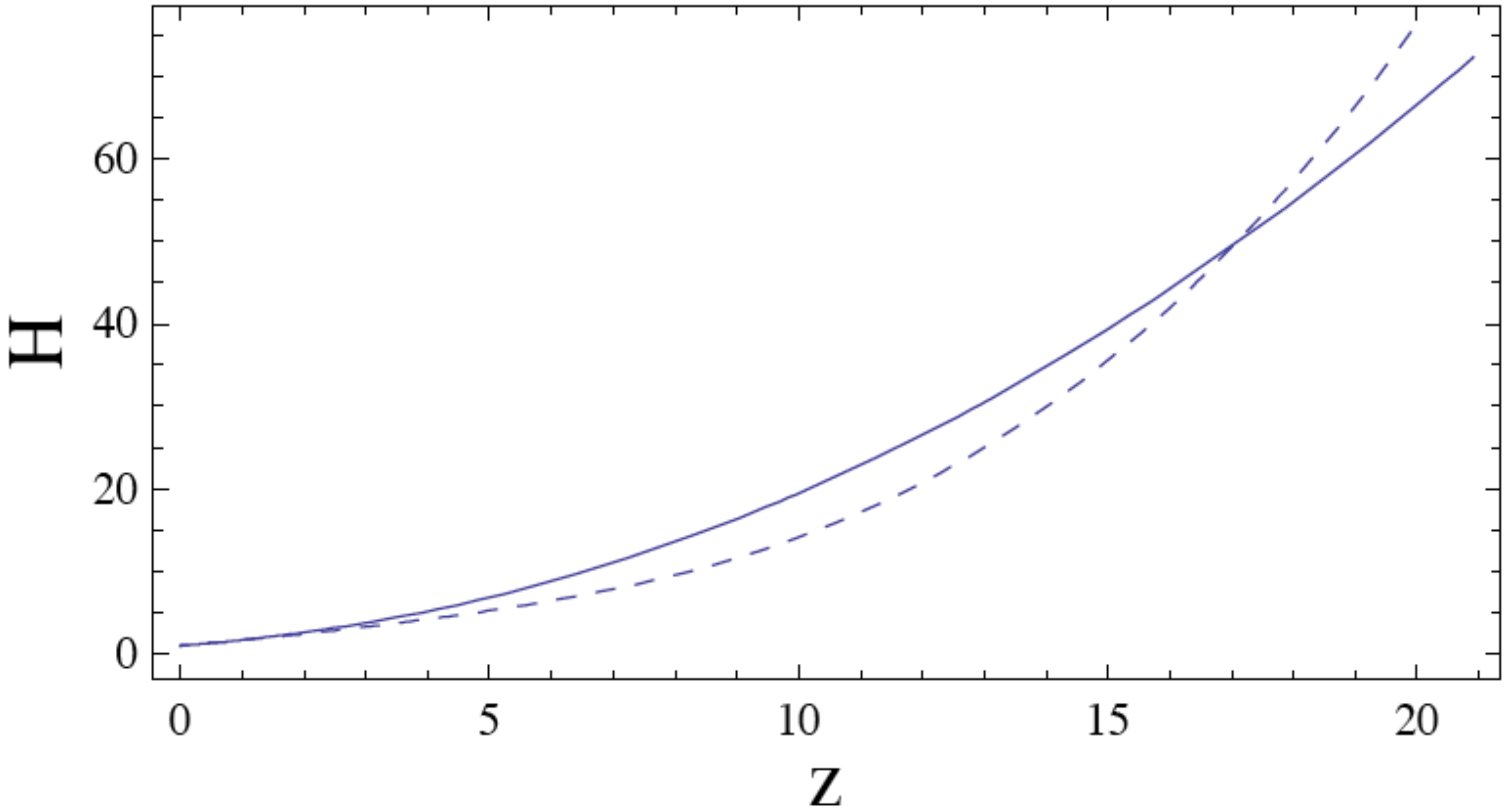
H is not a fiq, but aH is!



from S. Capozziello et. al., PLB 689 (2010) 117

Geodesics in Jordan/Einstein representations

In the Jordan frame:

$$\frac{d}{d\sigma}\left(\frac{dx^\alpha}{d\sigma}\right) + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} = 0.$$

In the Einstein frame:

$$\frac{d}{d\bar{s}}\left(\frac{dx^\alpha}{d\bar{s}}\right) + \bar{\Gamma}^\alpha_{\mu\nu}\frac{dx^\mu}{d\bar{s}}\frac{dx^\nu}{d\bar{s}} + \frac{\partial_\mu\Omega}{\Omega}\frac{dx^\mu}{d\bar{s}}\frac{dx^\alpha}{d\bar{s}} = 0,$$

By proper re-parametrize the affine parameter and employing the relation between the Christoffel symbol in the two frames, it is straightforward to show that the geodesic equation becomes

$$\frac{d}{d\bar{\sigma}}\left(\frac{dx^\alpha}{d\bar{\sigma}}\right) + \bar{\Gamma}^\alpha_{\mu\nu}\frac{dx^\mu}{d\bar{\sigma}}\frac{dx^\nu}{d\bar{\sigma}} = 0.$$

Quiros et. al. arXiv:1108.5857

Summary

- Conformal transformation is merely the rescaling of units, not the transformation of coordinates
- JF and EF are indeed physically equivalence, or physically indistinguishable, at least in the regime of classical gravity
- One must carefully select the corresponding physical quantities to compare within these two frame